# Was Loitsyansky correct? A review of the arguments

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# Was Loitsyansky correct? A review of the arguments

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**Abstract.** Loitsyansky's integral, $I$, is important because it controls the rate of decay of kinetic energy in freely-evolving, isotropic turbulence. Traditionally it was assumed that $I$ is conserved in decaying turbulence and this leads to Kolmogorov's decay law, $u^2 \sim t^{-10/7}$. However, the modern consensus is that $I$ is not conserved, which is a little surprising since Kolmogorov's law is reasonably in line with the experimental data. This discrepancy led Davidson (2000 *J. Fluid Mech.* submitted) to reassess the entire problem. He concluded that, for certain initial conditions, which are probably typical of wind tunnel turbulence, freely evolving turbulence reaches an asymptotic state in which the variation of $I$ is negligible, a conclusion which is at odds with the predictions of certain closure models. In this review we revisit this debate. We explain why the widespread belief in the time dependence of $I$ owes much to a misinterpretation of Batchelor and Proudman's original analysis (1956 *Phil. Trans. R. Soc.* A **248** 369). Indeed, a survey of the experimental and numerical data shows that there is little evidence for significant long-range pressure forces of the type which underpin the supposed variation of $I$. Interestingly, Batchelor and Proudman reached the same conclusion almost half a century ago. We conclude by extending the ideas of Loitsyansky and Kolmogorov to MHD turbulence. We note that there exists a Loitsyansky integral for MHD turbulence (Davidson 1997 *J. Fluid Mech.* **336** 123) and show that this leads to energy decay laws which coincide with the experimental data.

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## Contents

## 1. Introduction

The dynamics of the largest eddies in freely evolving turbulence has been the source of much controversy for almost half a century. The debate was initiated by Batchelor and Proudman [3] who were the first to note that significant statistical correlations may exist between remote points in a turbulent flow. The information is communicated over large distances by the pressure field which is, of course, non-local. That is, the relationship

$$\nabla^2 p = -\rho \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} \tag{1.1}$$

may be inverted using the Biot–Savart law to give

$$p(\boldsymbol{x}) = \frac{\rho}{4\pi} \int \frac{\partial^2 u_i'' u_j''}{\partial x_i'' x_j''} \frac{\mathrm{d}\boldsymbol{x}''}{|\boldsymbol{x}'' - \boldsymbol{x}|}. \tag{1.2}$$

Thus, a fluctuation in velocity at one point, say $\boldsymbol{x}'$, sends out pressure waves which propagate to all parts of the flow field. Suppose, for example, that we have a single eddy located near $\boldsymbol{x} = \boldsymbol{0}$. Then the pressure field at large distances from the eddy, due to that eddy, is

$$p = \frac{\rho}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{x} \right) \int u_i'' u_j'' \, \mathrm{d}\boldsymbol{x}.$$

(The first two terms in the Taylor expansion of $|\boldsymbol{x}'' - \boldsymbol{x}|^{-1}$ lead to integrands which integrate to zero.) Thus the pressure field associated with our eddy falls off rather slowly, in fact as $x^{-3}$. This pressure field induces forces, and hence motion, throughout the fluid and of course this motion is correlated to the behaviour of the eddy at $\boldsymbol{x} = \boldsymbol{0}$.

     Batchelor and Proudman showed that, in homogeneous turbulence which is anisotropic, these non-local effects can give rise to long-range pressure–velocity correlations of the form

$$\langle u_i u_j p' \rangle_\infty \sim r^{-3} \qquad \boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}.$$

These, in turn, induce long-range velocity correlations like $\langle uuu' \rangle_\infty \sim r^{-4}$, $\langle uu' \rangle_\infty \sim r^{-5}$. This discovery has profound ramifications for the way in which we view the large scales in a turbulent

flow. It implies that every point in the flow senses every other point, and that remote boundaries could influence the bulk properties of the turbulence.

Up until 1956 it had been believed that remote points in a turbulent flow are statistically independent in the sense that $\langle uu' \rangle$ is exponentially small at large $r$. This change in viewpoint undermined the classical theories of large-scale dynamics, particularly those theories which had been developed by Kolmogorov, Landau and Loitsyansky. For example, it was shown by Loitsyansky and Landau that, provided that remote points in a turbulence flow are statically independent, then

$$I = - \int r^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} = 8\pi u^2 \int_0^\infty r^4 f \, \mathrm{d}r \qquad (1.3)$$

is an invariant of decaying, isotropic turbulence. (Here $f$ is the longitudinal correlation function and $u^2 = \overline{u_x^2}$.) Loitsyansky's proof rested on integrating the Karman–Howarth equation,

$$\frac{\partial}{\partial t}[u^2 r^4 f] = u^3 \frac{\partial}{\partial r}[r^4 K] + 2vu^2 \frac{\partial}{\partial r}[r^4 f'(r)] \qquad (1.4)$$

while Landau's derivation was characteristically more physical. In particular, he showed that the angular momentum of a cloud of turbulence evolving in a large spherical domain is related to $I$ by

$$I = \langle \boldsymbol{H}^2 \rangle / V + 0(l/R) \qquad R \gg 1. \qquad (1.5)$$

(Here $\boldsymbol{H}$ is the global angular momentum of the fluid, $\int \boldsymbol{x} \times \boldsymbol{u} \, \mathrm{d}V$, $R$ is the radius of the domain, $V = 4/3\pi R^3$, and $l$ is the integral scale of the turbulence.) Thus, conservation of angular momentum for each realization ensures conservation of $I$.

Batchelor's discovery of long-range correlations appears to invalidate both of these derivations. In the case of Loitsyansky's proof the contribution to $\mathrm{d}I/\mathrm{d}t$ from the triple longitudinal correlation function, $8\pi u^3 \lfloor r^4 K \rfloor_\infty$, is assumed to vanish, while Landau had assumed that the anisotropy introduced by the boundary $|\boldsymbol{x}| = R$ is confined to a vanishingly small percentage of $V$ as $R/l \to \infty$. Both of these assumptions become questionable when we admit the possibility of long-range correlations.

Now the question of whether or not $I$ is an invariant is not just an esoteric matter. In the first place, it calls into question Kolmogorov's decay law,

$$u^2 \sim t^{-10/7} \qquad (1.6)$$

which comes from integrating the energy decay equation

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} \sim -\frac{u^3}{l} \qquad (1.7)$$

subject to the constraint

$$I \sim u^2 l^5 = \text{constant}. \qquad (1.8)$$

Second, the alleged invariance of $I$ is responsible for the phenomenon of the *permanence of the big eddies*. That is, for many types of turbulence the energy spectrum grows as $k^4$ at small $k$ and in such cases

$$E(k) = (I/4!\pi^2)k^4 + \cdots. \qquad (1.9)$$

Thus, in the absence of long-range pressure forces, $E(k)$ is of fixed shape for small $k$.

This uncertainty over the behaviour of $I$ came at an interesting time in the development of turbulence theory. Closure models based on the quasi-normal approximation were beginning to gain ground and, as we shall see, these predict that $I$ is time dependent. Indeed, it was Proudman and Reid's [10] paper on the quasi-normal hypothesis which first drew Batchelor's attention to

the possibility that $I$ is time dependent. Given the predictions of the closure models, and the persuasive arguments of Batchelor and Proudman in the context of anisotropic turbulence, a consensus emerged that the old views were flawed and that $I$ is time dependent on isotropic turbulence. This view still prevails.

However, in some ways, this is a little surprising. Consider Landau's argument. The only assumption made by Landau is that the bulk properties of a turbulent flow evolving in an extremely large domain are independent of the details of the remote boundaries. This is in accordance with our intuition. For example, we do not believe that the lateral boundaries in a large wind-tunnel greatly influence the behaviour of grid turbulence. Nor do we believe that the symmetry planes in a periodic cube unduly pollute the bulk properties of an (almost) isotropic turbulent flow provided, of course, that $L_{BOX} \gg l$†.

It is the apparent discrepancy between our intuition and the prevailing view, that $I$ is time dependent, which makes this such an interesting topic. Recently, Davidson [6] revisited this problem in the context of isotropic turbulence. He concluded that, for certain initial conditions, which are probably typical of wind tunnel turbulence, decaying turbulence reaches an asymptotic state in which the variation of $I$ is negligible. That is, once the details of the initial conditions are largely forgotten, the rate of change of $I$ is two orders of magnitude smaller than that predicted by the quasi-normal hypothesis and so, for most practical purposes, $I$ may be treated as a constant.

In this paper we review the now familiar arguments in favour of $I$ being time dependent and then summarize the counter arguments of Davidson [6]. The structure of the paper is as follows. In section 2 we discuss the classical (pre-Batchelor and Proudman) theories of Landau, Loitsyansky and Kolmogorov. Then, in section 3, we discuss the celebrated attacks on the classical view. There were three deadly blows arising from the work of Proudman and Reid [10], Batchelor and Proudman [3] and Saffman [11]. In sections 4 and 5 we summarize the arguments of Davidson [6] which includes: (i) a reappraisal of the three key objections to the classical view; (ii) a discussion of the significance of the closure model predictions of $I(t)$; and (iii) a review of the experimental and numerical data. We conclude, in section 6, with a discussion of MHD turbulence. Here we show that the ideas of Landau and Kolmogorov may be extended to include turbulent, conducting fluids evolving in a uniform magnetic field. This yields decay laws which are analogous to Kolmogorov's law and which are consistent with the experimental evidence.

Throughout we are concerned with turbulence which is initially isotropic (or nearly isotropic) and which is of the type which might be generated in a wind tunnel.

## 2. The classical theories of Landau, Loitsyansky and Kolmogorov

The Landau–Loitsyansky equation for isotropic turbulence

$$I = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} = \langle \boldsymbol{H}^2 \rangle / V = \text{constant} \tag{2.1}$$

lies at the centre of the controversy and so it is worth summarizing the assumptions implicit in (2.1). As noted in section 1, Loitsyansky's approach to the problem was to integrate the

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† The symmetry planes not only impose anisotropy at the large scales, but also enforce unphysical long-range correlations at the scale of $L_{BOX}$. In order for the turbulence in a periodic cube to evolve in a manner which is more or less independent of the presence of the symmetry planes it is generally agreed that $L_{BOX}/l$ must be large. Failure to meet this criterion can result in unphysical behaviour. For example, under-resolved situations can show an $E \sim k^4$ spectrum spontaneously converting into a $E \sim k^2$ spectrum, something which cannot happen in homogeneous isotropic turbulence.

Karman–Howarth equation (1.4). Provided that $K$ decreases at least as fast as $r^{-5}$ at large $r$ this yields

$$u^2 \int_0^\infty r^4 f \, \mathrm{d}r = I/8\pi = \text{constant.} \tag{2.2}$$

Traditionally it had been assumed that remote points in a turbulent flow are statistically independent in the sense that $f$ and $K$ decline transcendentally fast as $r \to \infty$. Thus, the assumption made by Loitsyansky seemed, at the time, entirely reasonable and, as a consequence (2.2) was, for some time, assumed to be rigorous. This was given tentative support by the fact that Kolmogorov's decay law $u^2 \sim t^{-10/7}$, which is based on (2.2), seems not too far out of line with the experimental data. These results were generalized by Batchelor who showed that, in anisotropic turbulence, integrals of the type $\int r_m r_n \langle u_i u_j' \rangle \, \mathrm{d}r$ are invariants.

The physical significance of the conservation of $I$ was first clarified by Landau. He dispensed with the Karman–Howarth equation and considered instead the angular momentum, $\boldsymbol{H}$, of a turbulent cloud evolving in a large sphere, $R \gg l$. Now the first point to note is that, for random initial conditions, $\boldsymbol{H}$ will not be exactly zero even though the sphere is large. Indeed, the central limit theorem tells us that, if $\boldsymbol{x} \times \boldsymbol{u}$ at each location can be considered as an independent random variable, then $\langle \boldsymbol{H}^2 \rangle$ should be proportional to $V$, the volume of the turbulent cloud. That is, there will always be some incomplete cancellation of the angular momentum of the randomly orientated eddies in the sphere. If we make the sphere larger and larger, then the average angular momentum density $[\langle \boldsymbol{H}^2 \rangle]^{1/2}/V$ does indeed tend to zero as we would expect. Nevertheless, $\langle \boldsymbol{H}^2 \rangle$ grows as $V$.

Now suppose that we stir up the contents of the sphere and then leave the turbulence to decay. We repeat the experiment many times and form ensemble averages of the turbulent quantities. Over many realizations the average of $\boldsymbol{H}$ will approach zero, $\langle \boldsymbol{H} \rangle = 0$. However, the average of $\boldsymbol{H}^2$ (a positive quantity) will be non-zero. In fact, it is readily confirmed that

$$\langle \boldsymbol{H}^2 \rangle = -\int\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}V. \tag{2.3}$$

(See Landau and Lifshitz [7].) Next Landau, like Loitsyansky, assumed that $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ decays rapidly with $\boldsymbol{r}$. In such a case only those velocity correlations taken close to the boundary are aware of the presence of this surface and in this sense the turbulence is approximately homogeneous and isotropic. Also, the far-field contributions to the inner integral in (2.3) are now small and so equation (2.3) reduces to

$$I = \langle \boldsymbol{H}^2 \rangle/V = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} + \mathrm{O}(l/R). \tag{2.4}$$

Now $\boldsymbol{H}$ need not be conserved in each realization. In fact,

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{T}_v$$

where $\boldsymbol{T}_v$ is the viscous torque exerted by the boundary on the fluid. However, the central limit theorem allows us to estimate the magnitude of $\boldsymbol{T}_v$ arising from randomly orientated eddies near the boundary. This suggests that $\boldsymbol{T}_v$ has negligible influence as $R/l \to \infty$. In this sense, then, $\boldsymbol{H}$ (and hence $\boldsymbol{H}^2$) is conserved in each realization and it follows that $I$ is an invariant.

The fact that the invariance of $I$ could be established by two distinct routes, and that Kolmogorov's decay law, which is based on the conservation of $I$, is reasonably in line with the experiments, meant that most people were, for some time, happy with (2.1). Everything changed, however, in 1956.

## 3. Three lethal blows to Loitsyansky's invariant

The first sign that all was not well with the Landau–Loitsyansky equation came with the work of Proudman and Reid [10] in isotropic turbulence. They investigated the dynamical consequences of a popular closure scheme called the quasi-normal approximation. (This closes the problem at third order by assuming that the velocity statistics are Gaussian to the extent that the cumulants of the fourth-order velocity correlations are zero. This allows the fourth-order correlations to be expressed as products of second-order correlations.) They found that, when the quasi-normal (QN) approximation is made, the triple correlations appear to decay as $r^{-4}$ at large $r$, rather than exponentially fast as had traditionally been assumed. If this were also true of real turbulence it would invalidate both Loitsyansky's and Landau's proofs of the invariance of $I$. Indeed the QN closure model predicts

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \frac{7}{5}(4\pi)^2 \int_0^\infty \frac{E^2}{k^2}\,\mathrm{d}k. \tag{3.1}$$

These findings were somewhat surprising and it provoked Batchelor and Proudman into revisiting the entire problem in 1956. They considered *anisotropic* turbulence and dispensed with the QN approximation. Their primary conclusion was that correlations of the form $\langle u_i u_j p'\rangle$ can decay rather slowly at large $r$, in fact as slowly as $r^{-3}$. This is a direct consequence of the slow decline in the pressure field ($p_\infty \sim r^{-3}$) associated with a local fluctuation in velocity (see equation (1.2)). Now the triple correlation

$$S_{ij,k}\langle u_i u_j u_k'\rangle$$

is governed by an equation of the form

$$\frac{\partial S_{ij,k}}{\partial t} = \langle uuuu'\rangle - \frac{\partial}{\partial r_k}\langle u_i u_j p'\rangle + \cdots \tag{3.2}$$

and so even if there is no algebraic tail in $S_{ij,k}$ at $t = 0$, there may well be for $t > 0$, and in general we would expect $\langle uuu'\rangle_\infty \sim r^{-4}$. It follows that, in anisotropic turbulence, we should expect $\langle uu'\rangle_\infty \sim r^{-5}$ and so generalized Loitsyansky integrals of the form

$$I_{ijmn} = \int r_m r_n \langle u_i u_j'\rangle\,\mathrm{d}\boldsymbol{r} \tag{3.3}$$

are only conditionally convergent. More importantly, the slow decline in $\langle uuu'\rangle$ means that $I_{ijmn}$ is, in general, time dependent and not an invariant as traditionally assumed. (See, for example, Batchelor [2] for the proof that, in the absence of long-range effects, $I_{ijmn}$ is an invariant.) Many concluded that, since $I_{ijmn}$ is time dependent in anisotropic turbulence, then $I$ should not be an invariant in isotropic turbulence.

The third blow to Loitsyansky's integral came with the work of Saffman [11], who noted that, *for certain initial conditions*, turbulence could sustain even stronger long-range correlations ($r^{-3}$ in $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$). Under these conditions Loitsyansky's integral diverges and the energy spectrum takes the form

$$E \sim Lk^2 + \cdots \tag{3.4}$$

where $L$ is proportional to the square of the linear momentum of the fluid,

$$L = \left\{\int \boldsymbol{u}\,\mathrm{d}V\right\}\Big/V = \int \langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r}. \tag{3.5}$$

The integral $L$ is an invariant of Saffman's spectrum because the total linear momentum of the turbulence is conserved. It is unaffected by the long-range pressure forces of Batchelor and Proudman as they contribute only to $\mathrm{O}(r^{-4})$ terms in $\langle uuu'\rangle_\infty$ and thus to the $k^4$ term in (3.4).

In conclusion then, Proudman and Reid [10] showed that a plausible closure model, which is kinematically admissible if dynamically groundless, leads to $\mathrm{d}^2 I / \mathrm{d} t^2$ being non-zero. Batchelor and Proudman [3] dispensed with the QN approximation and showed that, in general, one would expect $\langle uuu \rangle_\infty \sim r^{-4}$ in anisotropic turbulence with the consequence that $I_{ijmn}$ is time dependent. Finally, Saffman [11] showed that, for certain initial conditions, Loitsyansky's integral does not even exist! All-in-all, by the late 1960s, things looked bad for Loitsyansky's would-be invariant.

## 4. Weaknesses in the case against the invariance of $I$

Recall that we are interested in isotropic turbulence of the type generated in a wind tunnel. As noted by Davidson [6], a closer examination of the work by Proudman, Reid, Batchelor and Saffman shows that, for this type of turbulence, the case is less conclusive than one might have expected. Consider, for example, Batchelor and Proudman [3]. Although they established that $\langle uuu' \rangle$ decays as $r^{-4}$ in anisotropic turbulence, they were unable to find any net long-range pressure forces when the symmetries of isotropy are imposed. Indeed they concluded that the traditional assumption of an exponential decay of $\langle uu' \rangle_\infty$ and $\langle uuu' \rangle_\infty$ could not be ruled out. Moreover, they were aware that, in the final period of decay, measurements of $f(r)$ in grid turbulence indicates an exponential, rather than algebraic, decline in $\langle uu' \rangle$. This provoked them to note that:

> ... it is disconcerting that the present theory [i.e. their theory] cannot do as well as the old.

Now consider Saffman's objections. In order to obtain a Saffman spectrum in isotropic turbulence it is essential to ensure that $\langle uu' \rangle_\infty \sim r^{-3}$. Such a strong, long-range effect cannot arise from Batchelor's long-range pressure forces since, at most, they contribute only to a $K_\infty \sim r^{-4}$ term in (1.4). Thus, a Saffman spectrum can arise only if $f_\infty \sim r^{-3}$ at $t = 0$. It would seem, therefore, that the initial conditions determine whether or not a Batchelor ($E \sim k^4$) or a Saffman ($E \sim k^2$) spectrum emerges. Moreover, a Saffman spectrum requires that the global linear momentum grows as $[\int \boldsymbol{u} \, \mathrm{d} V]^2 \sim V$. For grid turbulence the indications are that the spectrum is of the Batchelor form ($E \sim k^4$), probably because the mechanism which generates the turbulence cannot impart sufficient linear momentum to the flow [11].

As for the work of Proudman and Reid, it is well known that the QN approximation is dynamically flawed, predicting anomalous results such as negative energy spectra. The most we can say from the QN model is that if, at $t = 0$, we specify Gaussian statistics for $\boldsymbol{u}$, then for an infinitesimal period $I$ will grow by an amount $(\delta t)^2$.

## 5. A reappraisal of the long-range pressure forces in isotropic turbulence

We now follow the arguments of Davidson [6]. The first thing to note is that there is an apparent discrepancy between Proudman and Reid [10] and Batchelor and Proudman [3]. The latter show that, for isotropic turbulence, the QN approximation yields

$$\frac{\mathrm{d}^2 I}{\mathrm{d} t^2} = 8\pi \frac{\mathrm{d}}{\mathrm{d} t}[u^3 r^4 K]_\infty = \frac{7}{5}(4\pi)^2 \int_0^\infty (E^2/k^2) \, \mathrm{d} k. \tag{5.1}$$

This requires that fourth-order cumulants of the form

$$[u_i u_j' u_k'' u_l]_c = \langle u_i u_j' u_k'' u_l \rangle - \langle u_i u_j' \rangle \langle u_k'' u_l \rangle - \langle u_i u_k'' \rangle \langle u_j' u_l \rangle - \langle u_i u_l \rangle \langle u_j' u_k'' \rangle \tag{5.2}$$

are zero *for all sets of points* $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{x}''$ at all times. This is certainly not the case in real turbulence, but we can at least arrange for (5.2) to hold at $t = 0$ if not for $t > 0$. Thus we cannot rule out a $K_\infty \sim r^{-4}$ tail in isotropic turbulence for purely kinematic reasons.

Compare this with the findings of Batchelor and Proudman. They assumed that, at $t = 0$, $\langle uu' \rangle$, $\langle uuu' \rangle$ and $\lfloor u_i u'_j u''_k u_l \rfloor_c$ are all exponentially small for *well separated points*. This is much less restrictive than the QN approximation and is simply an assumption that, at $t = 0$, remote points are statistically independent. The question then is whether or not the pressure field will induce algebraic tails in $\langle uu' \rangle_\infty$ and $\langle uuu' \rangle_\infty$ for $t > 0$. For isotropic turbulence they find no reason why $\langle uuu' \rangle_\infty$ should not remain exponentially small, which seems to contradict (5.1).

A re-examination of the isotropic problem, in the spirit of Batchelor and Proudman, reveals that, if $s = u_x^2 - u_y^2$,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8\pi \frac{\mathrm{d}}{\mathrm{d}t}[u^3 r^4 K]_\infty = 6 \int \langle ss' \rangle \, \mathrm{d}\boldsymbol{r} = 6J \tag{5.3}$$

provided $\lfloor u_i u'_j u''_k u_l \rfloor_c$ is exponentially small for *well separated points* [6]. Now in mature turbulence (turbulence which has largely forgotten its initial conditions and has developed a full range of length scales) the experimental data suggests that the joint probability distribution of the velocity at two points becomes normal as $r \to \infty$. Thus, Davidson [6] took (5.3) to hold for turbulence in the asymptotic state. In fact, (5.1) is a special case of (5.3) since if we specify that $\lfloor u_i u'_j u''_k u_l \rfloor_c$ is zero for *all* $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{x}''$, then (5.3) yields

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 6J = \frac{14}{5} \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle^2 \, \mathrm{d}\boldsymbol{r} = \frac{7}{5}(4\pi)^2 \int_0^\infty (E^2/k^2) \, \mathrm{d}k.$$

Thus, as suggested by Proudman and Reid, there is indeed the possibility of $I$ varying with $t$ in isotropic turbulence. Of course, the key question is: what is the value of $J$ in the asymptotic state? Now $J$ is non-negative since

$$J = \int \langle ss' \rangle \, \mathrm{d}\boldsymbol{r} = \left\{ \int s \, \mathrm{d}V \right\}^2 \Big/ V. \tag{5.4}$$

Thus, following Davidson [6], we can write

$$J = \alpha u^4 L^4 \qquad \alpha \geq 0 \tag{5.5}$$

where the integral scale $L$ is defined via

$$I = u^2 L^5. \tag{5.6}$$

Now (5.3) in the form

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 6\alpha u^4 L^3 \tag{5.7}$$

can be integrated along with

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} \sim -\frac{u^3}{L} \tag{5.8}$$

to obtain estimates of $I(t)$ and $u^2(t)$. (It is necessary at this point to assume that $\alpha$ is constant in the asymptotic state.) A comparison of these predictions with the numerical and experimental data shows that $\alpha$ lies in the range 0–0.03, with a mean value of $\alpha \sim 0.01$. (Davidson [6] considered six independent sets of data.) The quasi-normal estimate of $\alpha$, on the other hand, gives $\alpha \sim 0.6$. Evidently, if there are long-range pressure forces of the type envisaged by Batchelor and Proudman, then they are very weak, two orders of magnitude smaller than that predicted by the QN closure scheme. Thus, it seems that if $I$ does indeed grow, then it grows only very slowly.

An example of the near-invariance of $I$ is shown in figure 1 which is taken from the large-eddy simulation (LES) of Lesieur *et al* [8]. Although there is some initial variation in $I$, it
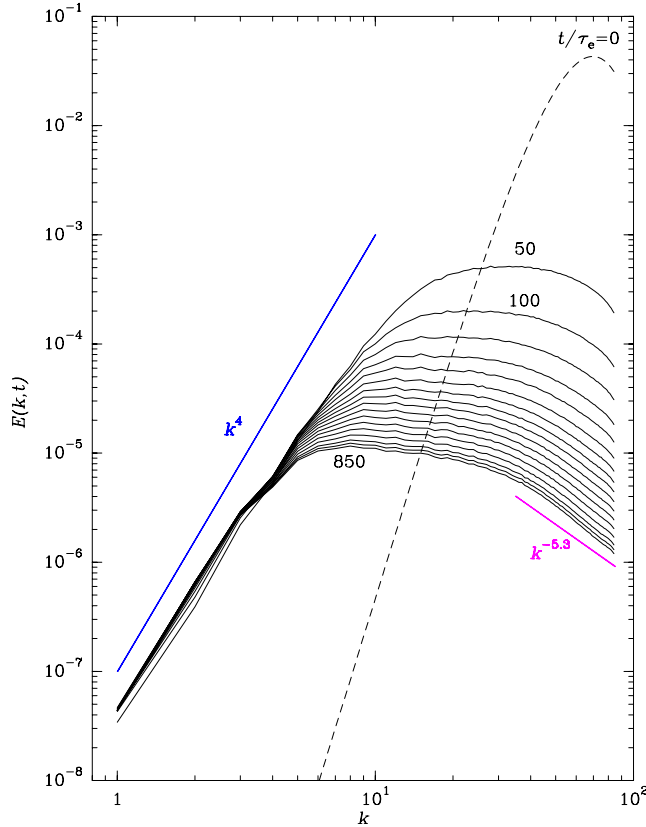
**Figure 1.** Energy spectra in decaying turbulence calculated by Lesieur *et al* [8].

settles down to a constant value in the asymptotic state. All in all, it seems that the classical view is not too far from the truth. For all practical purposes we may treat $I$ as constant in the asymptotic phase. Why then, is there near universal agreement that $I$ is time dependent? Part of the reason for this may be the over-reliance that some researchers place on various closure models. Two typical models are the QN and EDQNM closure schemes. (The latter is a variant of the QN model in which certain *ad hoc* alterations are made to the QN equations in order to avoid negative energy spectra.) These schemes make the following claims:

(i)  QN:  $$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 6J_{QN} = \frac{14}{5} \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle^2 \, \mathrm{d}\boldsymbol{r} \sim 4u^4 L^3 \tag{5.9}$$

(ii)  EDQNM:  $$\frac{\mathrm{d}I}{\mathrm{d}t} = 8\pi [u^3 r^4 K]_\infty \sim \theta J_{QN} \tag{5.10}$$

where $\theta$ is a somewhat arbitrary parameter with the dimensions of time. (Typically $\theta$ is related to the turn-over time of the small eddies.) However, we have already seen that the QN estimate is greatly out of line with the experimental data, while the arbitrary decision to remove a time derivative (compare (5.3) and (5.10)) in the EDQNM scheme seems hard to justify. In summary, then, we have no reliable means of estimating the magnitude of the long-range effects (only *ad hoc* closure schemes) and all we can say with certainty is that, for all practical purposes, they are negligible.

## 6. The decay of homogeneous turbulence in a magnetic field

We now turn to MHD turbulence. We shall see that many of the classical ideas of Landau and Kolmogorov carry over to MHD turbulence, with little modification, and that they produce predictions which are in line with the experiments.

### 6.1. From isotropic turbulence to MHD turbulence

Traditionally, MHD turbulence has been studied by two rather distinct communities. On the one hand, engineers have studied low-magnetic Reynolds number turbulence, motivated largely by the need to understand the flow of liquid metal in technological devices. Here, much attention has been focused on the influence of boundaries in, for example, duct flows. The Hartmann layer plays a central role in such theories. On the other hand, plasma physicists and astrophysicists tend to study turbulence at high-magnetic Reynolds numbers, $R_m = ul/\lambda \gg 1$. (Here $\lambda$ is the magnetic diffusivity.) Much of this work has focused on homogeneous turbulence and the motivation is very often provided by the dynamics of the solar wind and solar corona and by dynamo theory.

  The purpose of this final section is to show that the ideas of Landau and Kolmogorov can be redeveloped in the context of MHD turbulence, and that this provides a unified view of freely-decaying, homogeneous turbulence, valid for arbitrary magnetic Reynolds number. In the limit of low $R_m$ we have in mind the need to characterize small-scale turbulence in the core of the earth, and also liquid-metal turbulence in the many metallurgical operations where the boundaries have little influence on the turbulence.

  Our starting point is the paper by Davidson [5] in which a Loitsyansky-like integral was established for low-$R_m$ turbulence. When the long-range correlations are weak, as they are in conventional turbulence, this integral becomes an invariant. We extend this earlier work in two ways. First we show that, in the absence of long-range correlations, the integral is also an invariant of high-$R_m$ turbulence. Second, we derive a decay law for low-$R_m$ turbulence which is analogous to Kolmogorov's law. This new law is compared with experiments and the two are found to coincide.

### 6.2. Landau revisited

Let us repeat Landau's thought experiment, adapted now to MHD turbulence. Suppose that a conducting fluid is held in a large, insulated sphere. The fluid is stirred up and then left to itself. This time, however, the sphere sits in a uniform, imposed magnetic field, $\boldsymbol{B}_0$, so that the turbulence is subject to a Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$. (Here $\boldsymbol{B}$ is the total magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$, $\boldsymbol{b}$ being associated with the currents, $\boldsymbol{J}$, induced by $\boldsymbol{u}$ within the sphere.) Since Landau's arguments relate to conservation of $\boldsymbol{H}$, we must evaluate the torques acting on the fluid. The global torque exerted on the fluid by the Lorentz force is

$$\boldsymbol{T}_B = \int \boldsymbol{x} \times (\boldsymbol{J} \times \boldsymbol{B}_0)\, \mathrm{d}V + \int \boldsymbol{x} \times (\boldsymbol{J} \times \boldsymbol{b})\, \mathrm{d}V. \tag{6.1}$$

However, the second integral on the right is zero since a closed system of currents interacting with its self-magnetic field cannot give rise to a net torque. (This follows from the conservation of angular momentum.) Also, the first integral can be transformed using the identity

$$2\boldsymbol{x} \times [\boldsymbol{G} \times \boldsymbol{B}_0] = [\boldsymbol{x} \times \boldsymbol{G}] \times \boldsymbol{B}_0 + \nabla \cdot [(\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{B}_0))\boldsymbol{G}] \tag{6.2}$$

where $\boldsymbol{G}$ is any solenoidal field. Combining (6.1) and (6.2) we have

$$\boldsymbol{T}_B = \left\{ \frac{1}{2} \int \boldsymbol{x} \times \boldsymbol{J}\, \mathrm{d}V \right\} \times \boldsymbol{B}_0 = \boldsymbol{m} \times \boldsymbol{B}_0 \tag{6.3}$$

where $\boldsymbol{m}$ is the dipole moment of the induced current. Evidently, the angular momentum of the fluid evolves according to

$$\rho\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{T}_v + \boldsymbol{T}_B = \boldsymbol{T}_v + \boldsymbol{m} \times \boldsymbol{B}_0. \tag{6.4}$$

Following Landau we note that, if $R \gg l$, the influence of $\boldsymbol{T}_v$, the viscous torque, will be negligible on the timescale of the decay. Thus,

$$\rho\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{m} \times \boldsymbol{B}_0. \tag{6.5}$$

It follows that $\boldsymbol{H}_\parallel$ is conserved. In the event that $R_m$ is low, we can also determine the variation of $\boldsymbol{H}_\perp$. That is, the low-$R_m$ version of Ohm's law,

$$\boldsymbol{J} = \sigma(-\nabla V + \boldsymbol{u} \times \boldsymbol{B}_0) \tag{6.6}$$

combined with (6.2) yields

$$\boldsymbol{m} = (\sigma/4)\boldsymbol{H} \times \boldsymbol{B}_0. \tag{6.7}$$

(Here $\sigma$ is the electrical conductivity.) It follows that

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = -\frac{\boldsymbol{H}_\perp}{4\tau} \qquad \tau^{-1} = \sigma B_0^2/\rho \tag{6.8}$$

where $\tau$ is known as the Joule damping time. Evidently, $\boldsymbol{H}_\parallel$ is conserved while $\boldsymbol{H}_\perp$ decays exponentially fast on a time scale of $4\tau$.

Let us now return to the general case of arbitrary $R_m$. It is not difficult to show that

$$\langle\boldsymbol{H}_\parallel^2\rangle = -\iint \boldsymbol{r}_\perp^2 \langle\boldsymbol{u}_\perp \cdot \boldsymbol{u}_\perp'\rangle \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}\boldsymbol{x}$$

where $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$. If we now ignore Batchelor's long-range pressure forces we have, in the spirit of Landau,

$$I_\parallel = \langle\boldsymbol{H}_\parallel^2\rangle/V = -\int \boldsymbol{r}_\perp^2 \langle\boldsymbol{u}_\perp \cdot \boldsymbol{u}_\perp'\rangle \, \mathrm{d}\boldsymbol{r} = \text{constant}. \tag{6.9}$$

This was first noted, in the context of low-$R_m$ turbulence, by Davidson [5].

As in conventional turbulence, this invariant may also be derived from the Karman–Howarth equation. The argument proceeds as follows: the equation of motion

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_k}[u_k u_i - b_k b_i/\rho\mu] - \frac{\partial}{\partial x_i}\left[\frac{p}{\rho}\right] + (\boldsymbol{J} \times \boldsymbol{B}_0)_i/\rho + v\nabla^2 u_i \tag{6.10}$$

yields the generalized Karman–Howarth equation,

$$\frac{\partial}{\partial t}\langle u_i u_j'\rangle = \frac{\partial}{\partial r_k}[\langle u_i u_k u_j' - b_i b_k u_j'/\rho\mu\rangle + \langle u_j u_k u_i' - b_j b_k u_i'/\rho\mu\rangle] + \frac{1}{\rho}\left[\frac{\partial}{\partial r_i}\langle p u_i'\rangle + \frac{\partial}{\partial r_j}\langle p u_i'\rangle\right]$$
$$+ 2v\nabla^2\langle u_i u_j\rangle\frac{1}{\rho}[\langle(\boldsymbol{J} \times \boldsymbol{B}_0)_i u_j' + (\boldsymbol{J}' \times \boldsymbol{B}_0)_j u_i\rangle]. \tag{6.11}$$

Consider first the case where $\boldsymbol{B}_0$ and $\boldsymbol{b}$ are both zero. Then, following the arguments of Batchelor [2] it is readily shown that (6.11) yields

$$I_{ijmn} = \int r_m r_n \langle u_i u_j\rangle \, \mathrm{d}\boldsymbol{r} = \text{constant} \tag{6.12}$$

provided, of course, that there are no long-range correlations. This is a generalization of Loitsyansky's integral. When $\boldsymbol{b}$ is finite, but $\boldsymbol{B}_0$ remains zero, Batchelor's arguments may be repeated and again we find that $I_{ijmn}$ is an invariant. This was first noted by Chandrasekhar [4] in the context of isotropic turbulence. Let us now turn to the case where $\boldsymbol{B}_0$ is finite. In the

absence of long-range correlations, only the final term in (6.11) can contribute to the rate of change of integrals of the type $I_{ijmn}$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \int r_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \, \mathrm{d}\boldsymbol{r} = \frac{1}{\rho} \int r_\perp^2 \langle (\boldsymbol{J} \times \boldsymbol{B}_0)_\perp \cdot \boldsymbol{u}'_\perp + (\boldsymbol{J}' \times \boldsymbol{B}_0)_\perp \cdot \boldsymbol{u}_\perp \rangle \, \mathrm{d}\boldsymbol{r}. \quad (6.13)$$

The integrand on the right-hand side consists of terms of the form $r_\perp^2 J_y u'_x$ and $r_\perp^2 J_x u'_y$. (We take $\boldsymbol{B}_0$ to point in the $z$ direction.) Such terms can be converted into surface integrals since

$$3x^2 u_x = \nabla \cdot (x^3 \boldsymbol{u}) \qquad 3x^3 J_x = \nabla \cdot (x^3 \boldsymbol{J}) \qquad \text{etc}$$

and it follows that, in the absence of long-range correlations,

$$I_\parallel = -\int \boldsymbol{r}_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \, \mathrm{d}\boldsymbol{r} = \text{constant}. \quad (6.14)$$

We have arrived back at (6.9), but by a different route. Of course, the Landau approach is to be preferred since it exposes the physical origin of the invariant (6.14).

### 6.3. Decay laws at low $R_m$

Let us now repeat Kolmogorov's arguments, adapted to low-$R_m$ MHD turbulence. Ohm's law (6.6) tells us that

$$\nabla \times \boldsymbol{J} = \sigma \boldsymbol{B}_0 \cdot \nabla \boldsymbol{u}$$

and it follows that the Joule dissipation can be estimated as

$$\frac{\langle \boldsymbol{J}^2 \rangle}{\rho \sigma} \sim \left( \frac{l_\perp}{l_\parallel} \right)^2 \frac{u^2}{\tau}$$

where $l_{min}$ and $l_\parallel$ are suitably defined integral scales. Now we know that the effect of $\boldsymbol{B}_0$ is to introduce anisotropy into the turbulence, with $l_\parallel > l_\perp$ (see Davidson [5]). Thus we have

$$\frac{\langle \boldsymbol{J}^2 \rangle}{\rho \sigma} = \frac{3\beta}{2} \left( \frac{l_{min}}{l_\parallel} \right)^2 \frac{u^2}{\tau} \quad (6.15)$$

where $\beta$ is of order unity. (In fact it can be shown that $\beta = 2/3$ when the turbulence is isotropic.) We can use (6.15) to estimate the rate of decay of kinetic energy. That is, the energy equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle \boldsymbol{u}^2 \rangle}{2} = -v \langle \omega^2 \rangle - \langle \boldsymbol{J}^2 \rangle / (\rho \sigma) \quad (6.16)$$

can be rewritten as

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\alpha \frac{u^3}{l_\perp} - \beta \left( \frac{l_\perp}{l_\parallel} \right)^2 \frac{u^2}{\tau}. \quad (6.17)$$

Here we have made the usual estimate of the viscous dissipation. (In conventional turbulence $\alpha$ is of the order of unity.) Now our energy equation might be combined with (6.14) in the form

$$u^2 l_\perp^4 l_\parallel = \text{constant} \quad (6.18)$$

which offers the possibility of predicting $u^2(t)$ as well as $l_\perp$ and $l_\parallel$. In low-$R_m$ turbulence it is conventional to categorize the flow according to the value of the interaction parameter,

$$N = \frac{\sigma B_0^2 l_\perp}{\rho u} = \frac{l_\perp / u}{\tau}. \quad (6.19)$$

When $N$ is small (negligible magnetic effects), (6.17) and (6.18) reduce to

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\alpha \frac{u^3}{l} \qquad u^2 l^5 = \text{constant}$$

which yields Kolmogorov's law,

$$u^2 \sim t^{-10/7}.$$

When $N$ is large, on the other hand, inertia is unimportant and we have

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\beta \left(\frac{l_\perp}{l_\parallel}\right)^2 \frac{u^2}{\tau} \qquad u^2 l_\perp^4 l_\parallel = \text{constant}.$$

It is known that $l_\perp$ remains constant during the decay of high-$N$ turbulence and in such a case our equations predict

$$u^2 \sim u_0^2 (t/\tau)^{-1/2} \tag{6.20}$$

$$l_\parallel = l_0[1 + 2\beta t/\tau]^{1/2} \qquad l_\perp = l_0 \tag{6.21}$$

which is known to be correct (see Davidson [5]). For intermediate values of $N$, however, we have a problem. Equations (6.17) and (6.18) between them contain three unknowns $u^2$, $l_\perp$ and $l_\parallel$. To close the system we might tentatively introduce the heuristic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{l_\parallel}{l_\perp}\right)^2 = \frac{2\beta}{\tau} \tag{6.22}$$

which has the merit of being exact for $N \to 0$ and $N \to \infty$ but cannot be justified for intermediate $N$. (Essentially the same equations were proposed by Widlund *et al* [12] in their one-point closure model of MHD turbulence.) Integrating (6.17), (6.18) and (6.22) yields

$$u^2/u_0^2 = \hat{t}^{-1/2}[1 + (7/15)(\hat{t}^{3/4} - 1)N_0^{-1}]^{-10/7} \tag{6.23}$$

$$l_\perp/l_0 = [1 + (7/15)(\hat{t}^{3/4} - 1)N_0^{-1}]^{2/7} \tag{6.24}$$

$$l_\parallel/l_\perp = \hat{t}^{1/2}[1 + (7/15)(\hat{t}^{3/4} - 1)N_0^{-1}]^{2/7} \tag{6.25}$$

where $N_0$ is the initial value of $N$ and $\hat{t} = 1 + 2(t/\tau)$. (For simplicity we have taken $\alpha = \beta = 1$.) The high- and low-$N$ results above are special cases of (6.23)–(6.25). For the case of $N_0 = 7/15$ we obtain a simple power law,

$$u^2/u_0^2 \sim \hat{t}^{-11/7} \qquad l_\parallel/l_0 \sim \hat{t}^{5/7} \tag{6.26}$$

and indeed these power laws are reasonable approximations to (6.23)–(6.25) for all values of $N_0$ around unity. The only experiments of low-$R_m$, homogeneous turbulence known to the author were carried out by Alemany *et al* [1] and they suggest $u^2 \sim t^{-1.6}$ for $N_0 \sim 1$. This compares favourably with (6.26).

### 6.4. A new invariant for high-$R_m$ turbulence

When the long-range correlations are weak, $I_\parallel$ is also an invariant of high-$R_m$ turbulence. We conclude by exploring the consequences of this. Now, for large $R_m$, the mean field $\boldsymbol{B}_0$ causes an equiportion of energy between $\boldsymbol{b}$ and $\boldsymbol{u}$. This is known as the Alfvén effect and arises because small-scale disturbances tend to convert their energy into Alfvén waves (see, for example, Oughton *et al* [9]). Thus, for large $R_m$, we have

$$u^2 l_\perp^4 l_\parallel = \text{constant} \tag{6.27}$$

$$\langle \boldsymbol{u}^2 \rangle \sim \langle \boldsymbol{b}^2 \rangle/(\rho\mu) \tag{6.28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\langle \boldsymbol{u}^2 \rangle}{2} + \frac{\langle \boldsymbol{b}^2 \rangle}{\rho\mu}\right] = -v\langle \omega^2 \rangle - \langle \boldsymbol{J}^2 \rangle/(\rho\mu). \tag{6.29}$$

These may be combined to yield

$$el_\perp^4 l_\parallel = \text{constant} \tag{6.30}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -v\langle\omega^2\rangle - \langle\boldsymbol{J}^2\rangle/(\rho\sigma) \tag{6.31}$$

where $e$ is the energy density. This suggests that, as $e$ falls, $l_\perp^4 l_\parallel$ must increase and the indications are that $l_\parallel$ grows faster than $l_\perp$, as in low-$R_m$ turbulence (see Oughton *et al* [9]). This concludes our survey of homogeneous MHD turbulence.

## 7. Conclusions

We have seen that, once isotropic turbulence has reached its asymptotic state, the classical theory of Landau and Loitsyansky provides a good approximation to the large-scale dynamics. We have adapted this theory to MHD turbulence and determined the MHD analogues of Kolmogorov's decay law. The predictions of this theory are in line with the experimental data.

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